CSE 8803RS: Recommendation Systems

Lecture 8: Matrix Factorization with Nuclear Norm

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Vector Norms

- For a vector $x = [x_1, \ldots, x_D]^T$,

$$\ell_0\text{-norm} : \|x\|_0 \equiv \#\{\text{non-zero elements of } x\}$$

$$\ell_1\text{-norm} : \|x\|_1 \equiv \sum_i |x_i|$$

$$\ell_2\text{-norm} : \|x\|_2 \equiv (\sum_i |x_i|^2)^{1/2}$$

Convex Relaxation

- Consider the sparse least squares problem,

$$\min_{x \in R^D} \|b - Ax\|_2^2 + \lambda \|x\|_0$$

  put a constraint on the number of nonzero entries of x

- Convex relaxation, LASSO

$$\min_{x \in R^D} \|b - Ax\|_2^2 + \lambda \|x\|_1$$

Matrix Norms

- For a matrix $A = [A_{ij}]$ with singular values $\sigma_1 \geq \sigma_2 \geq \dots$. Let $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_N]^T$, the vector of all singular values of A,

$$\text{Matrix 2-norm} : \|A\|_2 \equiv \sigma_1 = \|\sigma\|_\infty$$

$$\text{Frobenius norm} : \|A\|_F \equiv (\sum_{i,j} A_{ij}^2)^{1/2} = (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_N^2)^{1/2} = \|\sigma\|_2$$

$$\text{Matrix rank} : \text{rank}(A) \equiv \|\sigma\|_0$$

$$\text{Matrix nuclear norm} : \|A\|_* \equiv \sigma_1 + \sigma_2 + \cdots + \sigma_N = \|\sigma\|_1$$

Covex Relaxation for Rank Constraints

- Consider

$$E(U,V) = \frac{1}{2}\sum_{(i,j)\in O}(A_{ij} - \sum_{k=1}^{K} U_{ik}V_{jk})^2 + \frac{\lambda}{2}\sum_{i,k} U_{ik}^2 + \frac{\lambda}{2}\sum_{i,k} V_{jk}^2$$

- Let $B = UV^T$, then

$$E(U,V) = \frac{1}{2}\sum_{(i,j)\in O}(A_{ij} - B_{ij})^2 + \frac{\lambda}{2}(\|U\|_F^2 + \|V\|_F^2)$$

Covex Relaxation for Rank Constraints

- **Theorem.** The nuclear norm can be characterized as

$$\|B\|_* = \min_{B=UV} \|U\|_F \|V\|_F = \min_{B=UV} \frac{1}{2}(\|U\|_F^2 + \|V\|_F^2)$$

- The objective function $E(U, V)$ can be rexlaxed to

$$E(B) = \frac{1}{2} \sum_{(i,j)\in O} (A_{ij} - B_{ij})^2 + \lambda \|B\|_*$$

which is a convex function of B.

Proof of Theorem

- Let $X = UV^T$, and $X = PSQ^T$ is the SVD of X with $S = \mathrm{diag}(\sigma_1, \ldots, \sigma_N)$
- Let i-th row of $P^T U$ and $Q^T V$ be u_i and v_i, respectively. Then

$$\sigma_i = u_i v_i^T \leq \|u_i\|_2 \|v_i\|_2$$

and

$$\sigma_1 + \cdots + \sigma_N \leq \|u_1\|_2 \|v_1\|_2 + \cdots + \|u_N\|_2 \|v_N\|_2$$

- The result follows by noticing,

$$\sum_i \|u_i\|_2 \|v_i\|_2 \leq \left(\sum_i \|u_i\|_2^2 \right)^{1/2} \left(\sum_i \|v_i\|_2^2 \right)^{1/2} = \|U\|_F \|V\|_F$$

Solution for Complete Data Case

- Complete data case,

$$E(B) = \frac{1}{2}\|A - B\|_F^2 + \lambda\|B\|_*$$

- **Theorem.** The minimizer of $\min_B \; E(B)$ is given by

$$S_\lambda(A) = U\Sigma_\lambda V^T, \quad \Sigma_\lambda = \mathrm{diag}((\sigma_1 - \lambda)_+, (\sigma_2 - \lambda)_+, \dots)$$

where $A = U\Sigma V^T$ is the SVD of A, $x_+ = \max\{0, x\}$

Proof of Theorem

- Let the singular values of A and X be

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N \geq 0, \quad x_1 \geq x_2 \geq \cdots \geq x_N \geq 0,$$

respectively. Wieland-Hoffman theorem says

$$\|A - X\|_F^2 \geq \sum_i (\sigma_i - x_i)^2.$$

- Use Wieland-Hoffman theorem we can show

$$\|A - X\|_F^2 + \lambda \|X\|_* \geq \sum_{i=1}^N (\sigma_i - x_i)^2 + \lambda \sum_{i=1}^N x_i.$$

Proof of Theorem

- The right-hand side of the above is minimized when

$$x_i = (\sigma_i - \lambda_i)_+, \quad i = 1, \ldots, N,$$

where for any real number x, $x_+ = \max\{x, 0\}$.

- The minimal value for the right-hand side is

$$\lambda \sum_{i=1}^{r} \sigma_i + \sum_{i=r+1}^{N} \sigma_i^2,$$

where r is such that

$$\sigma_1 \geq \cdots \geq \sigma_r > \lambda > \sigma_{r+1} \geq \cdots \geq \sigma_N \geq 0.$$

- Then $X = U\mathrm{diag}((\sigma_1 - \lambda)_+, (\sigma_2 - \lambda)_+, \cdots, (\sigma_N - \lambda)_+)V^T$ is the optimal solution to the optimization problem

Algorithm: Incomplete Data

Mazumder, Hastie and Tibshirani

- Given A, decompose $A = P_O(A) + P_O^{\perp}(A)$
- Start with $B_0 = 0$, for $n = 0, 1, \ldots,$

$$B_{n+1} = S_{\lambda}(P_O(A) + P_O^{\perp}(B_n))$$

- Need to compute partial SVD of

$$P_O(A) + P_O^{\perp}(B_n) = \underbrace{(P_O(A) - P_O(B_n))}_{\text{Sparse}} + \underbrace{B_n}_{\text{Low}-\text{rank}}$$

using Lanczos-type of methods.

SVD via Lanczos Bidigonalization

- Bidiagonalization: dense matrices,

$$A = UBV^T, \quad B = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ & \alpha_2 & \beta_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \beta_{n-1} \\ & & & & \alpha_n \end{bmatrix}$$

- The above can be computed using Householder transformations (Golub-Kahan algorithm)
- Then QR algorithm applied to B reduces it to diagonal form

Golub-Kahan-Lanczos Bidigonalization

- From $A = UBV^T$,

$$AV = UB, \quad A^T U = VB^T$$

Consider the k-columns of both sides,

$$Av_k = \alpha_k u_k + \beta_{k-1} v_{k-1}, \quad A^T u_k = \alpha_k v_k + \beta_{k+1} v_{k+1}$$

or

$$\alpha_k u_k = Av_k - \beta_{k-1} v_{k-1}, \beta_{k+1} v_{k+1} = A^T u_k - \alpha_k v_k$$

and

$$\alpha_k = \|Av_k - \beta_{k-1} v_{k-1}\|_2, \quad \beta_{k+1} = \|A^T u_k - \alpha_k v_k\|_2$$

- Start with unit v_1 and $\beta_0 = 0$

Golub-Kahan-Lanczos Bidigonalization

- After k steps,

$$AV_k = U_k B_k, \quad A^T U_k = V_k B_k^T + \beta_{k+1} v_{k+1} e_k^T$$

- Compute the SVD of $B_k = P_k S_k Q_k^T$, singular values of S_k, approximate singular values of A
- $U_k P_k$ and $V_k Q_k$ give approximate singular vectors,

$$A \approx (U_k P_k) S_k (V_k Q_k)^T$$

- Computational bottleneck: matrix-vector multiplication with A and A^T
- Re-orthogonalization

Partial SVD by Random Projection/Sampling

- $A \in R^{m \times n}$ and a given ℓ
  1. Draw $\Omega \in R^{n \times \ell}$ iid standard Gaussian
  2. Form $Y = A\Omega \in R^{m \times \ell}$
  3. Compute an orthonormal basis Q of Y
  4. Compute $B = Q^T A$
  5. Compute the SVD of $B = U_B \Sigma V^T$
     — Through eigen-decomposition of BB^T for example
  6. Then $A \approx (UU_B)\Sigma V^T$

Partial SVD by Random Projection: Error Bounds

- Let $Y = A\Omega$ and P_Y orthogonal projection, $\ell = k + p$,

$$\mathcal{E}\|(I - P_Y)A\|_F \leq \left(1 + \frac{k}{p-1}\right)^{1/2} \|A - A_k\|_F$$

  where A_k best rank-k approximation of A.
- p is called oversampling factor